

Non-cyclic graphs of (non)orientable genus one

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Abstract

Let G be a finite non-cyclic group. The non-cyclic graph Γ_G of G is the graph whose vertex set is $G \setminus \text{Cyc}(G)$, two distinct vertices being adjacent if they do not generate a cyclic subgroup, where $\text{Cyc}(G) = \{a \in G : \langle a, b \rangle \text{ is cyclic for each } b \in G\}$. In this paper, we classify all finite non-cyclic groups G such that Γ_G has (non)orientable genus one.

Keywords: Non-cyclic graph, finite non-cyclic group, genus.

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1 Introduction

All graphs in this paper are undirected, with no loops or multiple edges. A graph Γ is called a *planar graph* if Γ can be drawn in the plane so that no two of its edges cross each other, and in this case we say that Γ can be embedded in the plane. For a non-planar graph, it can be embedded in some surface obtained from the sphere by attaching some handles or crosscaps. We denote by \mathbb{S}_k a sphere with k handles and by \mathbb{N}_k a sphere with k crosscaps. Note that both \mathbb{S}_0 and \mathbb{N}_0 are the sphere itself, and \mathbb{S}_1 and \mathbb{N}_1 are the torus and the projective plan, respectively. The smallest non-negative integer k such that a graph Γ can be embedded on \mathbb{S}_k (resp. \mathbb{N}_k) is called the *orientable genus* or *genus* (resp. *nonorientable genus*) of Γ , and is denoted by $\gamma(\Gamma)$ (resp. $\bar{\gamma}(\Gamma)$).

The problem of finding the graph genus is NP-hard [9]. The (non)orientable genera of some graphs constructed from some algebraic structures have been studied, for instance, see [3–5, 7, 10].

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All groups considered in this paper are finite. Denote by \mathbb{Z}_n and D_{2n} the cyclic group of order n and the dihedral group of order $2n$, respectively. Let G be a non-cyclic group. The *cyclicizer* $\text{Cyc}(G)$ of G is

$$\{a \in G : \langle a, b \rangle \text{ is cyclic for each } b \in G\}.$$

and is a normal subgroup of G (see [6]). The non-cyclic graph Γ_G of G is the graph whose vertex set is $G \setminus \text{Cyc}(G)$, and two distinct vertices being adjacent if they do not generate a cyclic subgroup. The non-cyclic graph Γ_G was first considered by Abdollahi and Hassanabadi [1] and they studied the properties of the graph and established some graph theoretical properties (such as regularity) of this graph in terms of the group ones. In [2], Abdollahi and Hassanabadi classified all non-cyclic groups G such that Γ_G is planar.

A natural question is the following: Which finite non-cyclic groups have their non-cyclic graphs have (non)orientable genus one? The goal of the paper is to find all non-cyclic graphs of (non)orientable genus one. Our main results are the following theorems.

Theorem 1.1. *Let G be a finite non-cyclic group. Then Γ_G has genus one if and only if G is isomorphic to one of the following groups:*

$$\mathbb{Z}_3^2, \mathbb{Z}_2^3, \mathbb{Z}_2 \times \mathbb{Z}_4, D_8, \mathbb{Z}_2 \times \mathbb{Z}_6. \quad (1)$$

Theorem 1.2. *Let G be a finite non-cyclic group. Then Γ_G has nonorientable genus one if and only if G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$ or D_8 .*

2 Preliminaries

An element of order 2 in a group is called an *involution*. Let G be a group and g be an element of G . Denote by $|G|$ and $|g|$ the orders of G and g , respectively. We denote the symmetric group on n letters and the quaternion group of order 8 by S_n and Q_8 , respectively. Also \mathbb{Z}_n^m is used for the m -fold direct product of the cyclic group \mathbb{Z}_n with itself. In the following, we state some results which we need in the sequel.

Lemma 2.1. ([2, Proposition 4.3]) *Γ_G is planar if and only if G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, S_3 or Q_8 .*

Let Γ be a graph. Denote by $V(\Gamma)$ and $E(\Gamma)$ the vertex set and the edge set of Γ , respectively. We use the notation $\lceil x \rceil$ to denote the least integer that is greater than or equal to x . Denote by K_n and $K_{m,n}$ the complete graph of order n and the complete bipartite graph, respectively. The following result from [11] gives the (non)orientable genus of a complete graph and a complete multipartite graph.

Lemma 2.2. ([11]) *Let n be an integer at least 3. Then*

- (a) $\gamma(K_n) = \lceil \frac{1}{12}(n-3)(n-4) \rceil$.
- (b) $\bar{\gamma}(K_n) = \lceil \frac{1}{6}(n-3)(n-4) \rceil$ if $n \neq 7$ and $\bar{\gamma}(K_7) = 3$.
- (c) $\gamma(K_{m,n}) = \lceil \frac{1}{4}(m-2)(n-2) \rceil$.
- (d) $\bar{\gamma}(K_{m,n}) = \lceil \frac{1}{2}(m-2)(n-2) \rceil$.
- (e) $\gamma(K_{n,n,n}) = \frac{1}{2}(n-1)(n-2)$.
- (f) $\gamma(K_{n,n,n,n}) = (n-1)^2$ for $n \neq 3$ and $\gamma(K_{3,3,3,3}) = 5$.

Lemma 2.3. ([8, pp. 252, Theorem 9.7.3]) *Suppose that G is a p -group for some prime p and has a unique subgroup of order p . If $p = 2$, then G is cyclic or generalized quaternion. If $p > 2$, then G is cyclic.*

The following result is one of Sylow theorems.

Theorem 2.4. *Suppose that G is a group and p is a prime divisor of $|G|$. Then the number of Sylow p -subgroups is congruent to 1 modulo p . In particular, the number of subgroups of order p is congruent to 1 modulo p .*

Denote by φ the Euler's totient function.

Lemma 2.5. *Let G be a non-cyclic group, p, q two distinct primes and m a positive integer at least 1. If $\gamma(\Gamma_G) = 1$, the each of the following statements does not hold:*

- (a) G has 4 cyclic subgroups of order p^m and an element of order q , where $\varphi(p^m) \geq 2$.
- (b) G has 4 cyclic subgroups of order 3 and $|G| \geq 10$.
- (c) G has 3 cyclic subgroups of order 4 and an element of order q^m , where $\varphi(q^m) \geq 3$ and $q \neq 2$.
- (d) G has 7 cyclic subgroups of order 2 and an element of order q , where $q \neq 2$.
- (e) G has 3 cyclic subgroups of order p^m , where $\varphi(p^m) \geq 4$.
- (f) G has 2 cyclic subgroups of order p^m , where $\varphi(p^m) \geq 5$.

Proof. (a) Suppose, for a contradiction, that (a) holds. Let $\langle a \rangle, \langle b \rangle, \langle c \rangle, \langle d \rangle$ be 4 cyclic subgroups of order p^m of G and g be an element of order q . If g and each element of $\{a, a^{-1}, b, b^{-1}, c, c^{-1}, d, d^{-1}\}$ cannot generate a cyclic subgroup, then the induced subgraph by $\{a, a^{-1}, b, b^{-1}, c, c^{-1}, d, d^{-1}, g\}$ has a subgraph isomorphic to $K_{4,5}$ that has partition sets $\{c, c^{-1}, d, d^{-1}\}$ and $\{a, a^{-1}, b, b^{-1}, g\}$ and so $\gamma(\Gamma_G) \geq \gamma(K_{4,5}) = 2$, a contradiction. Thus, we may suppose that g and an element of $\{a, a^{-1}, b, b^{-1}, c, c^{-1}, d, d^{-1}\}$ can generate a cyclic subgroup $\langle h \rangle$. Without loss of generality, let $\langle a, g \rangle = \langle h \rangle$. Then $h \in V(\Gamma_G)$ and thereby, the induced subgraph by $\{a, a^{-1}, b, b^{-1}, c, c^{-1}, d, d^{-1}, h\}$ has a subgraph isomorphic to $K_{4,5}$ that has partition

sets $\{c, c^{-1}, d, d^{-1}\}$ and $\{a, a^{-1}, b, b^{-1}, h\}$. So $\gamma(\Gamma_G) \geq \gamma(K_{4,5}) > 1$, also a contradiction.

It is similar to the proof of (a), we can prove (b), (c) and (d).

(e) Assume, to the contrary, that (e) holds. Take 4 generators in every cyclic subgroup of order p^m . Then it is easy to see that the induced subgraph by the generators has a subgraph isomorphic to $K_{4,4}$ that has genus 3 by Lemma 2.2, a contradiction.

(f) It is similar to the proof of (e). \square

Lemma 2.6. *Let G be a non-cyclic p -group, where p is a prime. Then $\gamma(\Gamma_G) = 1$ if and only if G is isomorphic to one of the following groups:*

$$\mathbb{Z}_3^2, \mathbb{Z}_2^3, \mathbb{Z}_2 \times \mathbb{Z}_4, D_8. \quad (2)$$

Proof. Note that $\Gamma_{\mathbb{Z}_3^2} \cong K_{2,2,2,2}$, $\Gamma_{\mathbb{Z}_2^3} \cong K_7$ and each of $\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_4}$ and Γ_{D_8} is a subgraph of K_7 . By Lemma 2.2, we see that Γ_G has genus one for each group G in (2). We next assume that $\gamma(\Gamma_G) = 1$.

Suppose that $p \geq 3$. Then, by Lemma 2.3 and Theorem 2.4 we have that G has at least 4 subgroups of order p . It follows from (e) and (b) of Lemma 2.5 that $|G| \leq 9$. This implies that $G \cong \mathbb{Z}_3^2$, as desired.

Now suppose that $p = 2$. If $|G| \leq 8$, then G is isomorphic to \mathbb{Z}_2^2 , Q_8 , \mathbb{Z}_2^3 , $\mathbb{Z}_2 \times \mathbb{Z}_4$ or D_8 and by Lemma 2.1 we get the desired result. Thus, we may suppose that $|G| \geq 16$. If G is generalized quaternion, then G has a subgroup $\langle x \rangle$ of order 8 and contains at least 4 elements y_1, \dots, y_4 of order 4 that do not belong to $\langle x \rangle$, and so Γ_G has a subgraph isomorphic to $K_{6,4}$ that has partition sets $\{x^i : 0 < i < 8, i \neq 4\}$ and $\{y_j : j = 1, \dots, 4\}$, a contradiction by Lemma 2.2. Therefore, by Lemma 2.3 we may assume that G has at least 3 involutions.

Case 1. G has an element g of order 8.

Suppose that G has two distinct subgroups $\langle g \rangle, \langle h \rangle$ of order 8. Then we may pick an involution a in $G \setminus (\langle g \rangle \cup \langle h \rangle)$. Now we get a subgraph of Γ_G isomorphic to $K_{5,4}$ that has partition sets $\{g, g^3, g^5, g^7, a\}$ and $\{h, h^3, h^5, h^7\}$, a contradiction by Lemma 2.2.

Thus, we may suppose that G has a unique subgroup $\langle u \rangle$ of order 8, which is normal in G . Take an involution b that does not belong to $\langle u \rangle$. Then $\langle u, b \rangle$ is a subgroup of order 16 and has precisely one subgroup of order 8. Since $b \notin \langle u \rangle$, $\langle u, b \rangle$ is not cyclic. Note that G is not generalized quaternion. By verifying the groups of order 16, we get that $G \cong D_{16}$ or QD_{16} , where $QD_{16} = \langle a, b : a^8 = b^2 = 1, bab = a^3 \rangle$. If $G \cong D_{16}$, then G has 9 involutions which induce a subgraph isomorphic to K_9 , $\gamma(\Gamma_G) \geq \gamma(K_9) = 3$ by Lemma 2.2, a contradiction. Note that QD_{16} has only 6

elements of order 4. If $G \cong QD_{16}$, then the subgraph induced by 6 elements of order 4 and 4 elements of order 8 has a subgraph isomorphic to $K_{6,4}$, also a contradiction.

Case 2. G has no elements of order 8.

If G has no elements of order 4, then Γ_G is isomorphic to $K_{|G|-1}$, a contradiction as $\gamma(K_{|G|-1}) > 1$ for $|G| \geq 16$. Thus, in this case we may assume that $\pi_e(G) = \{1, 2, 4\}$. Note that $|G| \geq 16$. Since all involutions induce a complete graph and $\gamma(K_8) \geq 2$, G has at least 8 elements of order 4. Since a power graph induced by 10 elements of order 4 of a group has a subgraph isomorphic to $K_{6,4}$ that has genus two, G has precisely 8 elements of order 4 and 7 involutions. Take an involution a in G that does not belong to any subgroup of order 4. Then it is easy to see that the subgraph induced by all elements of order 4 and a has a subgraph isomorphic to $K_{5,4}$ that has genus two, a contradiction. \square

3 Proof of the main theorems

Proof of Theorem 1.1. Note that $\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6}$ has a subgraph isomorphic to $K_{3,3}$. Hence $\gamma(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6}) \geq 1$. On the other hand, we can embed $\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6}$ into the tours as shown in Figure 1. This implies that $\gamma(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6}) = 1$. Now by Lemma 2.6 we see that Γ_G has

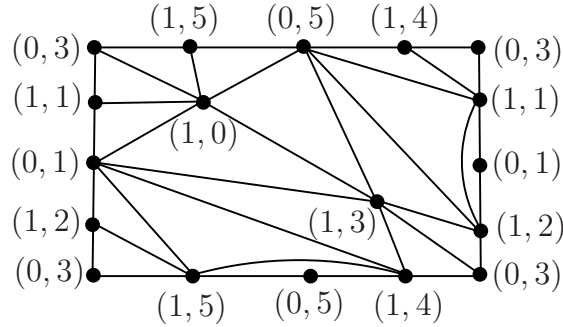


Figure 1: An embedding of $\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6}$ on \mathbb{S}_1 .

genus one for any group G in (1).

Now we assume that $\gamma(\Gamma_G) = 1$. If G is a p -group, the desired result follows from Lemma 2.6. Thus, we may assume that G is not a p -group. Let q be an odd prime divisor of $|G|$. If the number of subgroups of order q is not 1, then by Theorem 2.4 G has at least 4 subgroups of order q and by (a) of Lemma 2.5, we have a contradiction. Thus, G has a unique subgroup of order q and thereby, every Sylow q -subgroup of G is cyclic by Lemma 2.3. Similarly, we can get that G has a unique Sylow q -subgroup.

Note that G is not cyclic. Thus, we may assume that $G = P \rtimes Q$, where P is a 2-group and Q is a cyclic group of odd order. We next prove that P is not cyclic.

Suppose to the contrary that P is cyclic. Suppose that $|P| = 2$. Then G is dihedral. By Lemma 2.1, we see that $Q \not\cong \mathbb{Z}_3$ and so $\varphi(|Q|) \geq 4$ and G has at least 5 involutions. This implies that Γ_G has a subgraph isomorphic to $K_{5,4}$ that has two partition sets consisting of 5 involutions and 4 generators of Q , a contradiction. Suppose now that $P \cong \mathbb{Z}_4$. Note that G has at least 3 cyclic subgroups of order 4. By (c) of Lemma 2.5, we get $Q \cong \mathbb{Z}_3$. By checking the groups of order 12, $G \cong \langle a, b : a^6 = b^4 = 1, b^2 = a^3, b^{-1}ab = a^5 \rangle$. It is easy to check that $\gamma(\Gamma_G)$ has a subgraph isomorphic to $K_{4,5}$, a contradiction. If $|P| \geq 8$, since P is not normal in G , G has at least 3 Sylow 2-subgroups and since $\varphi(|P|) \geq 4$, a contradiction by (e) of Lemma 2.5. This means that P is not cyclic.

Note that $V(\Gamma_P) \subseteq V(\Gamma_G)$. Then $\gamma(\Gamma_P) = 0$ or 1 and by Lemmas 2.1 and 2.6, P is isomorphic to one of the following groups:

$$\mathbb{Z}_2^3, \mathbb{Z}_2 \times \mathbb{Z}_4, D_8, Q_8, \mathbb{Z}_2^2.$$

First by (d) of Lemma 2.5, we conclude $P \not\cong \mathbb{Z}_2^3$.

Case 1. $P \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

If P is not normal in G , then G has at least 4 cyclic subgroups of order 4, a contradiction by (a) of Lemma 2.5. Thus, $G \cong P \times Q$ and so G has precisely three involutions and at least two cyclic subgroups of order $4k$ for some odd prime k . Considering the generators of the two cyclic subgroups of order $4k$ and some involution, we have that Γ_G has a subgraph isomorphic to $K_{4,5}$, a contradiction.

Case 2. $P \cong D_8$.

If P is not normal in G , then G has at least 7 involutions in the union of all Sylow 2-subgroups, a contradiction by (d) of Lemma 2.5. Therefore, we may assume that $G \cong P \times Q$. Let g be an element of odd order. Then G has a cyclic subgroup of order $4|g|$, which has at least 4 generators $\{g_1, \dots, g_4\}$. Now it is easy to see that Γ_G has a subgraph isomorphic to $K_{4,5}$ that has two partition sets 4 involutions and $\{g_1, \dots, g_4, a\}$ for some involution a , a contradiction.

Case 3. $P \cong Q_8$.

Note that Q_8 has 3 cyclic subgroups of order 4. By (a) of Lemma 2.5, we may assume that $G \cong P \times Q$. So G has at least 3 cyclic subgroups of order $4k$ for some odd prime k . By (e) of Lemma 2.5, a contradiction.

Case 4. $P \cong \mathbb{Z}_2^2$.

If P is not normal in G , then G has at least 7 involutions, a contradiction by (d) of Lemma 2.5. Now we assume that $G \cong \mathbb{Z}_2^2 \times Q$. If $Q \cong \mathbb{Z}_3$, then as desired. Thus, we may assume that $|Q| \geq 5$. Then it is easy to see that G has at least 3 cyclic subgroups of order $2k$ for some odd number $k \neq 3$. By (e) of Lemma 2.5, a contradiction. \square

Proof of Theorem 1.2. Since $\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_4}$ and Γ_{D_8} all have some subgraphs isomorphic to $K_{3,3}$, one has that $\bar{\gamma}(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_4}) \geq 1$ and $\bar{\gamma}(\Gamma_{D_8}) \geq 1$. On the other hand, we may embed $\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_4}$ and Γ_{D_8} into \mathbb{N}_1 as shown in Figures 2 and 3, respectively. So we have $\bar{\gamma}(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_4}) = \bar{\gamma}(\Gamma_{D_8}) = 1$.

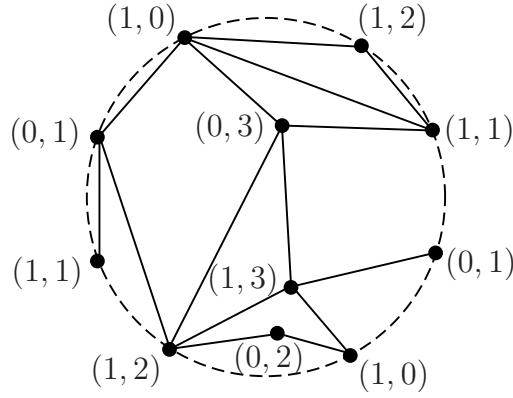


Figure 2: An embedding of $\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_4}$ on \mathbb{N}_1 .

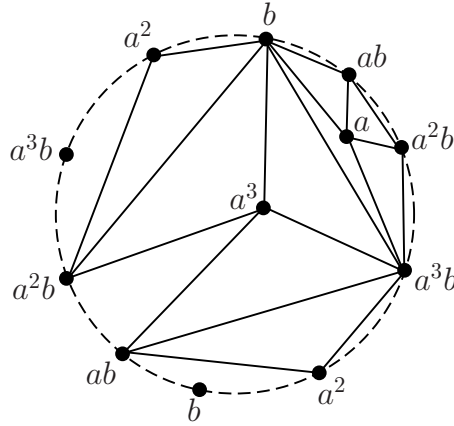


Figure 3: An embedding of Γ_{D_8} on \mathbb{N}_1 .

Now we assume that $\bar{\gamma}(\Gamma_G) = 1$. By Lemma 2.2 we see that $\bar{\gamma}(K_{4,4}) = 2$ and $\bar{\gamma}(K_n) \geq 2$ for $n \geq 7$. Thus, Γ_G has no subgraphs isomorphic to $K_{4,4}$ and K_n for

$n \geq 7$. By the proof of Theorem 1.1, it is easy to see that G is one group of (1). Since $\Gamma_{\mathbb{Z}_2^3}$ has a subgraph isomorphic to $K_{4,4}$ and $\Gamma_{\mathbb{Z}_2^3}$ has a subgraph isomorphic to K_7 , one has that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, D_8 or $\mathbb{Z}_2 \times \mathbb{Z}_6$. In order to complete our proof, we next prove $\bar{\gamma}(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6}) \geq 2$.

Clearly, $\bar{\gamma}(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6}) \geq 1$. Suppose for a contradiction that $\bar{\gamma}(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6}) = 1$. Note that $|V(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6})| = 9$ and $|E(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6})| = 27$. Thus, by the Euler characteristic formulas, if $\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6}$ is embedded into the surface of nonorientable genus $\bar{\gamma}(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6})$, resulting in f faces, then

$$|V(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6})| - |E(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6})| + f = 2 - \bar{\gamma}(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6}).$$

This implies that $2|E(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6})| \geq 3f$, which is a contradiction as $f = 19$. \square

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